On the Limiting Spectral Density Function of a Dynamic Nonconservative Birth-Death Q Matrix

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Outline

Background

Consider a stochastic system with mutually independent N elements (i.e. N bacterium), $X_1(t), X_2(t), ..., X_N(t)$, where t denotes the time. Assume that each element $X_k(t)$ $(1 \leq k \leq N)$ is a continuous-time birth-death Markov chain with the same birth rate b and death rate d. Let $S_N(t)$ denote the number of elements surviving in the system at time t. Then $\{S_N(t), t \geq 0\}$ is a time homogeneous Markov chain and its transition rate matrix (or Q matrix) Q_N can be written as

Background

$$Q_N = \begin{pmatrix} -bN & bN & & & & \\ d & -(d+b(N-1)) & b(N-1) & & & \\ & \ddots & \ddots & & & \\ & nd & -(nd+b(N-n)) & b(N-n) & & \\ & & \ddots & \ddots & & \\ & & (N-1)d & -((N-1)d+b) & b \\ & & & Nd & -Nd \end{pmatrix} \quad (1)$$

where $0 \leq n \leq N$.

Background

Naturally we may consider a general situation

$$Q_N = \begin{pmatrix} -r_N(0) & b_N(0) & & & & & \\ d_N(1) & -r_N(1) & b_N(1) & & & & \\ & \ddots & \ddots & & & & \\ & & d_N(n) & -r_N(n) & b_N(n) & & \\ & & & \ddots & \ddots & & \\ & & & d_N(N-1) & -r_N(N-1) & b_N(N-1) \\ & & & & d_N(N) & -r_N(N) \end{pmatrix} \qquad (2)$$

where $b_N(.) \geq 0$, $d_N(.) \geq 0$, $r_N(.) \geq 0$, are three nonnegative functions which depend on N, and $r_N(n)$ needs not equal to $b_N(n) + d_N(n)$ for $0 \leq n \leq N$ (Nonconservative !). We may call the above Q matrix Q_N a dynamic nonconservative birth-death Q matrix.

For example, let

$$b_N(n) = b(N-n)^\alpha, \quad d_N(n) = (n - N \ln \frac{n}{N})^\alpha$$
$$r_N(n) = b_N(n) + d_N(n) - c(N-n)^\alpha,$$

where $b \geq c$ and $\alpha \geq 0$.
It is clear that we can normalize the above Q_N matrix by multiplying $\frac{1}{N^\alpha}$, that is,

$$\frac{b_N(n)}{N^\alpha} = b(1 - \frac{n}{N})^\alpha, \quad \frac{d_N(n)}{N^\alpha} = (\frac{n}{N} - \ln \frac{n}{N})^\alpha$$
$$\frac{r_N(n)}{N^\alpha} = \frac{b_N(n) + d_N(n)}{N^\alpha} - c(1 - \frac{n}{N})^\alpha.$$

Obviously, when $n/N \to u$ $(0 < u < 1)$, we have

$$\frac{b_N(n)}{N^\alpha} \to b(1-u)^\alpha, \quad \frac{d_N(n)}{N^\alpha} \to d(u - \ln u)^\alpha$$
$$\frac{r_N(n)}{N^\alpha} \to (b-c)(1-u)^\alpha + d(u - \ln u)^\alpha.$$

Background

We recalled some known results.

- When $b_N(n) \equiv b > 0, d_N(n) \equiv d > 0, r_N(n) \equiv r \geq 0$, are three constants, Piet [Cambridge Univ. Press, 2011] gave a closed form of the limiting spectral density of Q_N.

Background

- When $b_N(n) = b(n), d_N(n) = d(n)$, $r_N(n) = r(n) \le b(n) + d(n))$, are three positive random variables for every $n \ge 0$ and the sequence $\{(b(n), d(n)), n \ge 0\}$ is i.i.d. or strictly stationary satisfying $E(b(n)/n^{\alpha})^k \to \mu_k$, $E(d(n)/n^{\alpha})^k \to \nu_k$ and $\sup_{n\ge 1} E(r^k(n)) < \infty$ for any $k \ge 1$, Popescu [Probab Theory Related Fields, 2009], Han and Zhang [Sci. Sin. Math. Chinese Ser., 2015] proved the existence and uniqueness of the limiting spectral distribution of the random birth-death Q matrices and gave the expression of the limiting spectral density function for some special cases ($\mu_k = \nu_k = 1$ for all $k \ge 1$).

Two Questions

- Under what conditions, there exits a unique limiting spectral density for the normalized dynamic nonconservative birth-death matrix $\bar{Q}_N$?
- What is the expression of the limiting spectral density function?

where $\bar{Q}_N = -N^{-\alpha} Q_N$ is the normalized matrix Q_N divided by N^{α} $(\alpha \geq 0)$.

The Main Results

By using the compatibility of the normalized dynamic nonconservative birth-death matrix $\bar{Q}_N = -N^{-\alpha} Q_N$ $(\alpha \geq 0)$, we know that the eigenvalues of $\bar{Q}_N$ are all real numbers. Denote them by $\lambda_0(N), \lambda_1(N), \ldots, \lambda_N(N)$.

Thus, the empirical spectral distribution $F_N(x)$ of the eigenvalues of $\bar{Q}_N = -N^{-\alpha} Q_N$ $(\alpha \geq 0)$ can be defined as

$$F_N(x) = \frac{1}{N+1} \sum_{k=0}^{N} I_{\{\lambda_k(N) \leq x\}}$$

where $I(\cdot)$ is the indicator function and x is any real number.

The Main Result

Theorem Let $b(u) > 0,\ d(u) > 0,\ r(u) \geq 0$ be three bounded continuous functions on $(0, 1)$ such that
$\frac{b_N(n)}{N^\alpha} = b(\frac{n}{N}) + o(1),\ \frac{d_N(n)}{N^\alpha} = d(\frac{n}{N}) + o(1),\ \frac{r_N(n)}{N^\alpha} = r(\frac{n}{N}) + o(1),$
for $1 \leq n \leq N-1$, where $o(1)$ denotes the infinitesimal for large N. Then there exists a unique limiting spectral distribution $F(x)$ of $\bar{Q}_N$

$$F(x) = \lim_{N\to\infty} F_N(x)$$

and the limiting spectral density function $p(x) = F'(x)$ has the following expression

$$p(x) = \frac{1}{\pi}\int_0^1 \frac{du}{\sqrt{4c(u) - (x - r(u))^2}} \qquad (3)$$

for $x \in [\alpha, \beta]$ and $p(x) = 0$ for $x \notin [\alpha, \beta]$, where $c(u) = b(u)d(u)$
$\alpha = \min_{0<u<1}\{r(u) - 2\sqrt{c(u)}\},\quad \beta = \max_{0<u<1}\{r(u) + 2\sqrt{c(u)}\}.$

The Main Result

Corollary Let $r(u) \equiv 0$ and the maximum value of $b(u)d(u)$ is a^2, that is, $a^2 = \max_{0<u<1} b(u)d(u)$. Then the limiting spectral density is an even function on $[-2a, 2a]$.

Applications of the Theorem

Example 1. Let $b_N(n) = (N-n)b, d_N(n) = nd$ and $r_N(n) = (N-n)b + nd$ for $0 \le n \le N$. It is clear that the Q matrix Q_N is conservative. Take $\alpha = 1$, the elements of the normalized Q matrix $\bar{Q}_N = -\frac{1}{N} Q_N$ can be written as

$$\frac{b_N(n)}{N} = (1-\frac{n}{N})b,\ \frac{d_N(n)}{N} = (\frac{n}{N})d,\ \frac{r_N(n)}{N} = (\frac{n}{N})d + (1-\frac{n}{N})b,$$

$$r_N(0) = b_N(0) = Nb,\ r_N(N) = d_N(N) = Nd,$$

for $1 \le n \le N$. This means that $b(u) = (1-u)b,\ d(u) = ud$, $r(u) = ud + (1-u)b$ for $0 < u < 1$, where, $b, d > 0$.

Applications of the Theorem

By using the theorem, we know that there exists a unique limiting spectral density of $\bar{Q}_N$ and it can be written as

$$
\begin{aligned}
p(x) &= \frac{1}{\pi}\int_0^1 \frac{du}{\sqrt{4u(1-u)bd - (x-(ud+(1-u)b))^2}} \\
&= \frac{1}{\pi}\int_0^1 \frac{du}{(b+d)\sqrt{\frac{4x(b+d-x)bd}{(b+d)^4} - [u - \frac{b(b+d)+x(d-b)}{(b+d)^2}]^2}},
\end{aligned}
$$

for $x \in [0, b+d]$. Here, we can check that $\alpha = 0$ and $\beta = b+d$.

In order to obtain a closed form of the limiting spectral density $p(x)$, let

$$a^2 = \frac{4x(b+d-x)bd}{(b+d)^4}, \quad c = \frac{b(b+d)+x(d-b)}{(b+d)^2}.$$

Since $(t-c)^2 \leq a^2$, it follows that $c-a \leq t \leq a+c$ and $c-a \geq 0, \ a+c \leq 1$.

Applications of the Theorem

Thus

$$
\begin{aligned}
p(x) &= \frac{1}{\pi}\int_0^1 \frac{dt}{(b+d)\sqrt{a^2-(t-c)^2}} \\
&= \frac{1}{\pi}\int_{c-a}^{c+a} \frac{dt}{(b+d)\sqrt{a^2-(t-c)^2}} \\
&= \frac{1}{(b+d)}
\end{aligned}
$$

This means that the limiting spectral distribution of $\bar{Q}_N = -\frac{1}{N}Q_N$ is uniformly distributed over the interval $[0, b+d]$.

Applications of the Theorem

Example 2. Taking $r_N(n) \equiv 0$ $(0 \le n \le N)$ in Example 1, we know that $\bar{Q}_N = -\frac{1}{N} Q_N$ is a dynamic nonconservative birth-death Q matrix. By using the theorem, its limiting spectral density can be written as

$$\begin{aligned} p(x) &= \frac{1}{\pi} \int_0^1 \frac{du}{\sqrt{4bdu(1-u) - x^2}} \\ &= \frac{1}{2\pi\sqrt{bd}} \int_0^1 \frac{dt}{\sqrt{\frac{1}{4} - \frac{x^2}{4bd} - (u - \frac{1}{2})^2}} \end{aligned}$$

for $x \in [-\sqrt{bd}, \sqrt{bd}]$.

Applications of the Theorem

In order to obtain a closed form of the limiting spectral density $p(x)$, let

$$c^2 = \frac{1}{4} - (\frac{x}{2\sqrt{bd}})^2.$$

Since $c^2 \geq (t - \frac{1}{2})^2$, it follows that

$$0 \leq \frac{1}{2} - c \leq t \leq \frac{1}{2} + c \leq 1$$

Thus

$$p(x) = \frac{1}{2\pi\sqrt{bd}} \int_{\frac{1}{2}-c}^{\frac{1}{2}+c} \frac{du}{\sqrt{c^2 - (u - \frac{1}{2})^2}} = \frac{1}{2\sqrt{bd}}$$

for $|x| \leq \sqrt{bd}$. That is, the limiting spectral distribution of $\bar{Q}_N$ is uniform on the interval $[-\sqrt{bd}, \sqrt{bd}]$.

Applications of the Theorem

Example 3. Let $b_N(n) \equiv b > 0$, $d_N(n) \equiv d > 0$ and $r_N(n) \equiv r \geq 0$ for $0 \leq n \leq N$. Taking $\alpha = 0$, the normalized Q matrix $\bar{Q}_N = -Q_N$, where

$$\bar{Q}_N = \begin{pmatrix} r & -b & & & \\ -d & r & -b & & \\ & \ddots & \ddots & & \\ & & -d & r & -b \\ & & & -d & r \end{pmatrix}$$

Here, r may not be equal to $b+d$. (may be nonconservative).

Applications of the Theorem

By the theorem, the limiting spectral density of $\bar{Q}_N$ can be written as

$$p(x) = \frac{1}{\pi}\int_0^1 \frac{du}{\sqrt{4bd-(x-r)^2}} = \frac{1}{\pi}\frac{1}{\sqrt{4bd-(x-r)^2}}$$

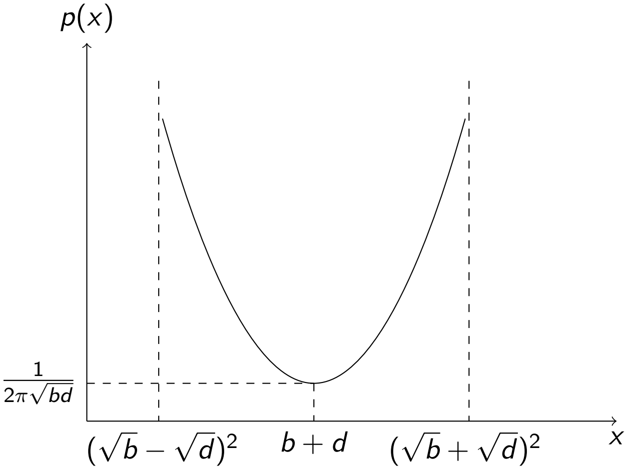
where $r-2\sqrt{bd} \leq x \leq r+2\sqrt{bd}$. This means that the minimum eigenvalue and the maximum eigenvalue of $\bar{Q}_N$ as $N \to \infty$ are $r-2\sqrt{bd}$ and $r+2\sqrt{bd}$ respectively. This is consistent with Piet's result [Cambridge Univ. Press, 2011].

Applications of the Theorem

Example 4. Taking $r_N(0) \equiv b, r_N(N) \equiv d$ and $r_N(n) \equiv b + d$ for $1 \le n < N$ in Example 3, we know that Q_N is a conservative Q matrix. Hence, 0 is an eigenvalue of $\bar{Q}_N = -Q_N$ and all other eigenvalues are great than or equal to 0. Since

$$(\sqrt{b} - \sqrt{d})^2 = r - 2\sqrt{bd} \le x \le r + 2\sqrt{bd} = (\sqrt{b} + \sqrt{d})^2$$

it follows that when $b \ne d$, the first nonzero eigenvalue (that is the second smallest eigenvalue) is $(\sqrt{b} - \sqrt{d})^2 > 0$ and the maximum eigenvalue is $(\sqrt{b} + \sqrt{d})^2$.


$p(x)$
$\frac{1}{2\pi\sqrt{bd}}$
$(\sqrt{b}-\sqrt{d})^2$
$b+d$
$(\sqrt{b}+\sqrt{d})^2$
x

The Main Steps of Proof

Step 1 Let $G_N(z)$ denote the characteristic function of the empirical spectral distribution $F_N(x)$ of the eigenvalues of $\bar{Q}_N = -N^{-\alpha} Q_N$ $(\alpha \geq 0)$. Then

$$\begin{aligned} G_N(z) &= \int e^{ixz} dF_N(x) = \frac{1}{N+1} \sum_{j=0}^{N} e^{i\lambda_j z} \\ &= \frac{1}{N+1} \sum_{j=0}^{N} \sum_{m=0}^{\infty} \frac{i^m \lambda_j^m z^m}{m!} \\ &= \frac{1}{N+1} \sum_{m=0}^{\infty} i^m Tr(\bar{Q}_N^m) \frac{z^m}{m!} \end{aligned} \tag{4}$$

The Main Steps of Proof

Step 2 Prove that $G(z) = \lim_{N\to\infty} G_N(z)$ exits.
Write $\bar{Q}_N$ as

$$\bar{Q}_N = Q_N(b,d) + D_N + o(1)$$

where $D_N = diag(r_N(0), r_N(\frac{1}{N}), \ldots, r_N(\frac{N-1}{N}), r_N(1))$,

$$\bar{Q}_N(b,d) = \begin{pmatrix} 0 & -b(0) & & & & \\ -d(\frac{1}{N}) & 0 & -b(\frac{1}{N}) & & & \\ & \ddots & \ddots & & & \\ & & -d(\frac{n}{N}) & 0 & -b(\frac{n}{N}) & \\ & & & \ddots & \ddots & \\ & & & & -d(\frac{N}{N}) & 0 \end{pmatrix}$$

The Main Steps of Proof

Hence

$$\bar{Q}_N^m = \sum_{k=0}^{m} C_m^k Q_N^k(b,d) D_N^{m-k} \tag{5}$$

Note that when k is odd number, $Tr(Q_N^k(b,d)) = 0$. Let

$$C_k^{\frac{k}{2}} = \begin{cases} 0, & \text{if } k \text{ is odd number,} \\ C_{2l}^l, & \text{if } k = 2l \text{ is even number.} \end{cases}$$

where C_{2l}^l is the combinatorial number.

The Main Steps of Proof

Hence

$$\frac{1}{N+1} Tr(Q_N^k(b,d) D_N^{m-k})$$
$$= \frac{1}{N+1} \sum_{n=1}^{N-1} C_k^{\frac{k}{2}} b^{\frac{k}{2}}(\frac{n}{N}) d^{\frac{k}{2}}(\frac{n}{N}) r^{m-k}(\frac{n}{N}) + O(\frac{1}{N+1}) \quad (6)$$

The Main Steps of Proof

By (4), (5), (6), we have

$$\begin{aligned} G(z) &= \lim_{N\to\infty} G_N(z) = \sum_{m=0}^{\infty} i^m \lim_{N\to\infty} \frac{1}{N+1} Tr(\bar{Q}_N^m) \frac{z^m}{m} \\ &= \sum_{m=0}^{\infty} i^m \left(\sum_{k=0}^{m} C_m^k \int_0^1 C_k^{\frac{k}{2}} b^{\frac{k}{2}}(u) d^{\frac{k}{2}}(u) r^{m-k}(u) du \right) \frac{z^m}{m!} \\ &= \int_0^1 \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!l!} \Big(\frac{2\sqrt{b(u)d(u)z}}{2}\Big)^{2l} \right) e^{ir(u)z} du \\ &= \int_0^1 J_0(2\sqrt{b(u)d(u)}z) e^{ir(u)z} dt \qquad (7) \end{aligned}$$

where $J_0(\cdot)$ is the first class of the modified Bessel function.

The Main Steps of Proof

Step 3 We prove that

$$\begin{aligned}\mu(m) &= \lim_{N\to\infty}\mu_N(m) = \lim_{N\to\infty}\int x^m dF_N(x) \\ &= \sum_{k=0}^{m} C_m^k C_k^{\frac{k}{2}} \int_0^1 b^{\frac{k}{2}}(u) d^{\frac{k}{2}}(u) r^{m-k}(u) du\end{aligned}$$

and

$$\sum_{m=0}^{\infty} \mu(2m)^{-\frac{1}{2m}} = +\infty$$

By using Carleman's theorem, we know that the sequence of moments $\{\mu(m); m \geq 0\}$ can uniquely determine the limiting spectral distribution function $F(x) = \lim_{N\to\infty} F_N(x)$.

The Main Steps of Proof

Step 4 We verify that the limiting spectral density function $p(x) = F'(x)$ exists and it can be written as

$$\begin{aligned} p(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-izx}G(z)dz \\ &= \frac{1}{2\pi}\int_0^1 dt\left[\int_{-\infty}^{+\infty} e^{-izx}J_0(2\sqrt{b(u)d(u)}z)e^{ir(u)z}dz\right] \\ &= \frac{1}{\pi}\int_0^1 \frac{du}{\sqrt{4b(u)d(u)-(x-r(u))^2}} \end{aligned}$$

- Prove the existence and uniqueness of the limiting spectral distribution of one kind of normalized dynamic nonconservative birth-death matrix $\bar{Q}_N$.
- Give an integral expression of the limiting spectral density function. We can get the closed form of the density function for some special cases.

References

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